Comparison of Uniformity Tests Against Non-Spherically Symmetric Alternative Hypotheses

Sun Yunze   (2220117)

Field of Intelligent Informatics, Department of Intelligent Systems
Supervisor: Associate Professor Kazunori Iwata

1 Introduction

This study examines uniformity tests on the unit disk. Conventional bivariate spherically symmetric distributions (SSDs) assume radial symmetry about the center of the disk. This assumption can be inflexible when data exhibit asymmetric skewness. We therefore propose a uniformity test based on the Möbius distribution (MD) [1] as an alternative.

2 Methods

For polar coordinates (r, θ) with $0 \le r \le 1$ and $-\pi \le \theta \le \pi$, the SSD density is

$$f(r; \gamma) = \frac{\gamma}{\pi} r \left(1 - r^2\right)^{\gamma - 1},$$

where γ controls concentration, with constraints $\gamma > 0$. We perform a uniformity test: $H_0 : \gamma = 1$ versus $H_1 : \gamma \ne 1$, using standard likelihood-ratio test (LRT) asymptotics. For polar coordinates (r, θ), the MD density

$$f(r, \theta; \gamma, a, \mu) = \frac{\gamma \left(1 - a^2\right)^{\gamma+1} r \left(1 - r^2\right)^{\gamma - 1}}{\pi \left(1 - 2ar\cos(\theta - \mu) + a^2 r^2\right)^{\gamma+1}}$$

has parameters controlling concentration, skewness, and orientation with constraints $\gamma > 0$, $0 \le a < 1$, and $-\pi < \mu \le \pi$. We test $H_0 : a = 0$ and $\gamma = 1$ versus $H_1 : a > 0$ or $\gamma \ne 1$. The LRT statistic is $2\ell(\hat{\gamma}, \hat{a}, \hat{\mu}) - 2\ell(0, 1, *)$, where ℓ is the MD log-likelihood, hats denote maximum-likelihood estimates (MLEs), and $*$ is an arbitrary orientation value. Because the null hypothesis H_0 places parameters on the boundary and orientation is not identifiable under H_0, standard LRT asymptotics do not apply in this case. We
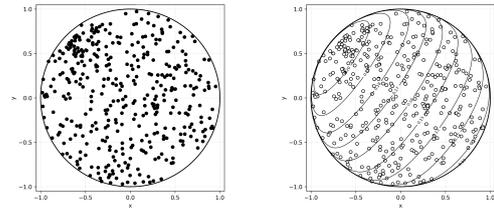
(a) Data points   (b) Density

Figure 1   Example of data points within the unit disk and fitted density.

therefore estimate p-values via a parametric bootstrap under H_0: resample, recompute the LRT, and evaluate the empirical distribution.

3 Experiments

Figure 1 shows data on the unit disk exhibiting asymmetric skewness, along with the fitted density. The MLEs were $(\hat{\gamma}, \hat{a}, \hat{\mu}) = (1.03, 0.08, 2.53)$, and the number of bootstrap replications was 2,000. Uniformity-test p-values were 0.68 (SSD) and 0.02 (MD); for reference, Watson's U test yielded 0.75. Thus, the MD-based uniformity test demonstrates greater power for asymmetrically skewed data.

4 Conclusion

We have presented a uniformity test based on the Möbius distribution for the unit disk. The parametric-bootstrap LRT has addressed boundary parameters and non-identifiable orientation, providing a practical test under asymmetric skewness.

References

[1] M. C. Jones, "The Möbius distribution on the disc," Ann. Inst. Statist. Math., 56(4), pp. 733–742, 2004.